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Ramsey problems for tilings in graphs

(Extended abstract)

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Abstract

Given a graph H, the Ramsey number $R(H)$ is the smallest $n \in \mathbb{N}$ such that every 2-edge-colouring of K_n yields a monochromatic copy of H. We write mH to denote the union of m vertex-disjoint copies of H. These graphs are also known as H-tilings. A famous result of Burr, Erdős and Spencer states that $R(mK_3) = 5m$ for $m \geq 2$. On the other hand, Moon proved that every 2-edge-coloured K_{3m+2} yields a mK_3 where each copy of K_3 is monochromatic, for $m \geq 2$. Crucially, in Moon's result, distinct copies of K_3 might receive different colours.

We investigate the analogous questions where the complete host graph is replaced by a graph of large minimum degree. We determine the largest size of a monochromatic K_3-tiling one can guarantee in any 2-edge-coloured graph of large minimum degree. We also determine the (asymptotic) minimum degree threshold for forcing a K_3-tiling covering a prescribed proportion of the vertices in a 2-edge-coloured graph such that every copy of K_3 in the tiling is monochromatic. These results therefore provide dense generalisations of the theorems of Burr–Erdős–Spencer and Moon.

1 Introduction

Ramsey theory is a central research topic in combinatorics. Ramsey's original theorem [17] asserts that for every graph H, there exists an $n \in \mathbb{N}$ such that every 2-edge-colouring of the complete graph K_n on n vertices yields a monochromatic copy of H. We write $R(H)$ to denote the smallest n for which the above holds.

In general, determining $R(H)$ is a very difficult problem and there are relatively few graphs H for which the exact value of $R(H)$ is known. An interesting class of graphs whose

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Ramsey behaviour is quite well-understood are so-called tilings. For a fixed graph H, an H-*tiling* is a collection of vertex-disjoint copies of H. For $m \in \mathbb{N}$, we write mH to denote an H-tiling consisting of m copies of H. The following result of Burr, Erdős and Spencer [5] determines the exact value of $R(mK_3)$.

Theorem 1.1 (Burr, Erdős and Spencer [5]). *We have $R(mK_3) = 5m$ for every $m \geq 2$.*

More generally, Burr, Erdős and Spencer [5] proved that, for a fixed graph H without isolated vertices, there exist constants c and m_0 such that $R(mH) = (2|H| - \alpha(H))m + c$ provided $m \geq m_0$, where $\alpha(H)$ is the independence number of H. Burr [4], and subsequently Bucić and Sudakov [3], provided methods for computing c exactly. Bucić and Sudakov [3] also obtained the current best bounds for m_0.

Although not a Ramsey-type question in the classical sense, it is also natural to ask how large a complete 2-edge-coloured graph needs to be to ensure there exists an H-tiling of a given size such that every copy of H is monochromatic. Crucially, in this setting, different copies of H in the tiling are allowed to receive different colours. The following result of Moon [15] settles the $H = K_3$ case of this problem (and was later generalised by Burr, Erdős and Spencer [5] to larger cliques).

Theorem 1.2 (Moon [15]). *For every integer $m \geq 2$, every 2-edge-colouring of K_{3m+2} yields a K_3-tiling consisting of m monochromatic copies of K_3. Furthermore, the term $3m+2$ cannot be replaced by a smaller integer.*

Schelp [18] (see also [14]) proposed the study of Ramsey-type questions where the host graph, rather than being complete, can be any graph satisfying a given minimum degree condition. Various results have been proven in this direction, see for example [1, 8, 9, 13]. Motivated by this line of research, in this extended abstract we consider the natural generalisations of the aforementioned classical Ramsey-type results about tilings to the dense setting. The works of Burr–Erdős–Spencer and Moon suggest the following two problems.

Problem 1.3. *Let H be a fixed graph and $n, r, \delta \in \mathbb{N}$. Determine the largest $m \in \mathbb{N}$ such that any r-edge-coloured n-vertex graph G with minimum degree $\delta(G) \geq \delta$ contains a monochromatic copy of mH.*

Problem 1.4. *Let H be a fixed graph and $n, r, \delta \in \mathbb{N}$. Determine the largest $m \in \mathbb{N}$ such that any r-edge-coloured n-vertex graph G with minimum degree $\delta(G) \geq \delta$ contains an H-tiling consisting of m monochromatic copies of H (and distinct copies of H in the tiling might be coloured differently).*

Various special cases of Problems 1.3 and 1.4 have already been considered and resolved. For example, the $r = 1$ case of both Problems 1.3 and 1.4 is equivalent to determining the largest H-tiling one can guarantee in any n-vertex graph G with $\delta(G) \geq \delta$. An H-tiling in a graph G is *perfect* if it contains all the vertices of G. The minimum degree threshold to force a perfect H-tiling in a graph was determined for $H = K_3$ by Corrádi and Hajnal [7], for $H = K_\ell$ (for any $\ell \in \mathbb{N}$) by Hajnal and Szemerédi [10] and for an arbitrary fixed graph H by Kühn and Osthus [12]. Komlós [11] determined (asymptotically) the minimum degree threshold that guarantees the existence of an H-tiling covering a fixed proportion of the vertices of the host graph, provided the proportion is less than 1, for any fixed graph H. Therefore, the $r = 1$ case of Problems 1.3 and 1.4 is (asymptotically) fully understood.

The $H = K_2$ case of both Problems 1.3 and 1.4 has also been resolved. The case $H = K_2$ of Problem 1.4 is equivalent to determining the largest K_2-tiling in a graph with given minimum degree, and thus it is covered by, for example, the Hajnal–Szemerédi theorem. The case $H = K_2$ of Problem 1.3 has a more interesting history. Given graphs $H_1, \dots, H_r$, we write $R_r(H_1, \dots, H_r)$ to denote the smallest integer n such that any r-edge-colouring of K_n using colours $c_1, \dots, c_r$ yields a monochromatic copy of H_i in colour c_i, for some i. Generalising a result of Cockayne and Lorimer [6], Gyárfás and Sárközy [9] determined $R_3(mK_2, mK_2, S_t)$ for all $t, m \in \mathbb{N}$, where S_t is the star on $t+1$ vertices. The connection of this purely Ramsey-type result to Problem 1.3 is that a red/blue/green edge-coloured K_n which does not contain a green monochromatic copy of S_t can be seen as a red/blue edge-coloured n-vertex graph G with $\delta(G) \geq n - t$. Therefore, Gyárfás and Sárközy's result resolves the case $H = K_2$, $r = 2$ of Problem 1.3. Omidi, Raeisi and Rahimi [16] computed $R_r(mK_2, \dots, mK_2, S_t)$ for all $r, t, m \in \mathbb{N}$, thus resolving the case $H = K_2$ of Problem 1.3 in full.

2 Main results

In this extended abstract, our main focus is to study Problems 1.3 and 1.4 when $H = K_3$ and $r = 2$. Observe that the case $\delta \leq 4n/5$ is uninteresting, as one cannot guarantee a single monochromatic copy of K_3. Indeed, consider a 2-edge-coloured K_5 that does not contain a monochromatic copy of K_3 and blow it up to obtain a 2-edge-coloured balanced complete 5-partite graph G on n vertices. Then $\delta(G) = \lfloor 4n/5 \rfloor$ and G does not contain a monochromatic copy of K_3. For Problem 1.3, the following theorem provides an exact answer when δ is a bit larger than $4n/5$ or a bit smaller than $n-1$.

Theorem 2.1. *Let $n \in \mathbb{N}$ and G be a 2-edge-coloured n-vertex graph. Then G contains a monochromatic copy of mK_3 where m is equal to*

(B.1) $\lfloor (\delta(G)+1)/5 \rfloor$ *if* $\frac{65n}{66} \leq \delta(G)$,

(B.2) $\lceil (5\delta(G) - 4n)/2 \rceil$ *if* $\frac{4n}{5} \leq \delta(G) \leq \frac{5n}{6}$.

Furthermore, parts (B.1) and (B.2) are best possible, in the sense that the statement of the theorem does not hold if m is replaced by a larger number.

The analogous case of Problem 1.4 turns out to be much more tractable. The following theorem provides an (asymptotic) resolution for all values of δ.

Theorem 2.2. *Let $n \in \mathbb{N}$ and G be a 2-edge-coloured n-vertex graph. Then there exists a K_3-tiling in G such that every copy of K_3 is monochromatic and the number of copies of K_3 in the tiling is at least*

(M.1) $\lfloor (2\delta(G) - n)/3 \rfloor$ *if* $\frac{7n}{8} \leq \delta(G)$,

(M.2) $\lfloor (4\delta(G) - 3n)/2 \rfloor - o(n)$ *if* $\frac{5n}{6} \leq \delta(G) \leq \frac{7n}{8}$,

(M.3) $5\delta(G) - 4n$ *if* $\frac{4n}{5} \leq \delta(G) \leq \frac{5n}{6}$.

Furthermore, parts (M.1) and (M.3) are best possible and part (M.2) is best possible up to the $o(n)$ term.

Note that Theorems 2.1 and 2.2 can be seen as dense generalisations of the results of Burr–Erdős–Spencer and Moon. Indeed, cases (B.1) and (M.1) imply Theorems 1.1 and Theorem 1.2 respectively.

The following constructions show the sharpness of case (B.1) of Theorem 2.1 and all cases of Theorem 2.2. For brevity, we omit the construction for case (B.2) of Theorem 2.1.

Construction for Theorem 2.1 (B.1). Let $n, \delta \in \mathbb{N}$ with $5 \leq \delta \leq n-1$. Let G be the n-vertex graph where all edges are present except for an independent set S of size $n - \delta \geq 1$. In particular, $\delta(G) = \delta$. Pick a partition $V(G) \setminus S = R \dot{\cup} B$ such that $|R| \leq 3\lfloor(\delta+1)/5\rfloor + 2$ and $|B| \leq 2\lfloor(\delta+1)/5\rfloor + 1$. Assign colour red to all edges that either lie in R or are between S and B. Assign colour blue to all edges that either lie in B or are between R and $S \cup B$.

Observe that there is no monochromatic K_3 intersecting S, i.e., every monochromatic copy of K_3 must lie in $R \cup B$. In particular, a red copy of K_3 must lie completely in R, while a blue copy of K_3 must have at least two vertices in B. Therefore, if there is a monochromatic mK_3 in G then $m \leq \max\{\lfloor |R|/3 \rfloor, \lfloor |B|/2 \rfloor\} \leq \lfloor \frac{\delta+1}{5} \rfloor$, as required.

Construction for Theorem 2.2. Let $n, \delta \in \mathbb{N}$ such that $4n/5 \leq \delta \leq n-1$. Let G be the following n-vertex graph. We have a partition $V(G) = V_0 \dot{\cup} V_1 \dot{\cup} \cdots \dot{\cup} V_5$ where $|V_i| = n - \delta \geq 1$ for every $i \geq 1$ and $|V_0| = 5\delta - 4n \geq 0$. The sets $V_1, \ldots, V_5$ are independent; all other pairs of vertices form an edge. It is easy to check that $\delta(G) = \delta$. Next, assign colours red and blue to the edges of G as follows. The subgraph $G[V_0 \cup V_1, V_2, V_3, V_4, V_5]$ is a blow-up of a red/blue edge-coloured K_5 that does not contain a monochromatic K_3. Without loss of generality, we may assume that the edges between $V_0 \cup V_1$ and $V_2 \cup V_3$ are blue while the edges between V_2 and V_3 are red. Finally, all edges lying in $V_0 \cup V_1$ are red.

By construction, the subgraph $G[V_0 \cup V_1, V_2, V_3, V_4, V_5]$ does not contain a monochromatic copy of K_3. It follows that every monochromatic copy of K_3 contains an edge lying in $V_0 \cup V_1$, and thus it must be red. In particular, every monochromatic copy of K_3 (i) has at least one vertex in V_0 (since V_1 is independent) and (ii) at least two vertices in $V_0 \cup V_1$. Furthermore, we have that (iii) no red monochromatic copy of K_3 intersects $V_2 \cup V_3$. If there are m vertex-disjoint monochromatic copies of K_3 in G, properties (i), (ii) and (iii) imply that $m \leq \min\{|V_0|, |V_0 \cup V_1|/2, (n - |V_2 \cup V_3|)/3\} = \min\{5\delta - 4n, (4\delta - 3n)/2, (2\delta - n)/3\}$.

3 Proof sketch of Theorems 2.1 and 2.2

For the proof of Theorem 2.2, we employ a common strategy for all cases (M.1)–(M.3): we first find many vertex-disjoint (blow-ups of) cliques in the host graph by combining the Hajnal–Szemerédi theorem [10] with the regularity method,[1] and then find monochromatic vertex-disjoint triangles within each such subgraph. This yields a large K_3-tiling where every copy of K_3 is monochromatic. Note also that case (B.2) of Theorem 2.1 follows immediately from case (M.3) of Theorem 2.2 and the pigeonhole principle. We do not provide further details on these arguments, and instead focus on the more subtle proof of case (B.1) of Theorem 2.1.

A *bowtie* consists of two monochromatic copies of K_3 of different colours which share exactly one vertex. The notion of a bowtie in this context was introduced by Burr, Erdős and Spencer [5], and played a crucial role in their proof of Theorem 1.1. The following new result is a key ingredient of the proof of Theorem 2.1 (B.1).

[1]Formally, the latter is only used in case (M2).

Lemma 3.1. *Suppose a 2-edge-coloured K_7 contains a bowtie. Then there exists another bowtie on a different vertex set.*

The proof of case (B.1) of Theorem 2.1 follows a vertex-switching type argument. Given an n-vertex graph G with $\delta(G) \geq \frac{65n}{66}$, we start by selecting a maximum collection $\mathcal{B}$ of vertex-disjoint copies of K_5, each containing a bowtie. Subject to this, we let $\mathcal{T}$ be a maximum collection of monochromatic triangles, all of the same colour, which are vertex-disjoint from each other as well as from the elements of $\mathcal{B}$. For brevity, we consider the case $\mathcal{T} \neq \emptyset$. Observe that G contains a monochromatic copy of $(|\mathcal{B}| + |\mathcal{T}|)K_3$, and so we may assume for a contradiction that $|\mathcal{B}| + |\mathcal{T}| < \lfloor (\delta(G) + 1)/5 \rfloor$. It follows that strictly less than $\delta(G)$ vertices lie in $\mathcal{B} \cup \mathcal{T}$ and thus there exists an edge e which is not incident to any element in $\mathcal{B} \cup \mathcal{T}$.

Let $T \in \mathcal{T}$ and set $X := T$. By using the fact that $\delta(G) \geq \frac{65n}{66}$, one can argue that there is some element B in $\mathcal{B}$ such that both $B \cup \{e\}$ and $B \cup X$ span two complete subgraphs. In particular, $B \cup \{e\}$ spans a copy of K_7 and so by Lemma 3.1 there exists a bowtie B' and a vertex $x \in V(B)$ such that $V(B') \subseteq V(B \cup \{e\})$ and $x \notin V(B')$. We then modify the collection $\mathcal{B}$ by removing B and adding B' in its place. Crucially, after this modification, the vertex x does not belong to $\mathcal{B}$ anymore and it is adjacent to all vertices of X (since $B \cup X$ spans a clique). We then add x to X. By iterating this procedure, we are able to increase the size of X while ensuring it still spans a clique. Note that the collection $\mathcal{T}$ is not affected at all in this process.

Once X reaches a sufficiently large size, by Moon's result (Theorem 1.2) it must contain two disjoint monochromatic triangles. If these have the same colour as the triangles in $\mathcal{T}$, we can add them to $\mathcal{T}$ in place of T, thus contradicting the maximality of $\mathcal{T}$. Otherwise, one can argue that X contains a bowtie, contradicting the maximality of $\mathcal{B}$.

Data availability statement. A full paper containing the proofs of our results can be found on arXiv [2].

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